# Functional integral approach for diffusion in a random media



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COMMENT

# Functional integral approach for diffusion in a random media

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**Abstract.** We present a local functional integral representation for the random diffusion equation studied by Kardar, Parisi and Zhang.

Diffusion in random media has been an important problem in several branches of theoretical and applied physics [1]. Our aim in this brief comment is to present a local functional integral solution for the random diffusion equation studied by Kardar, Parisi and Zhang [2].

Let us start our analysis by writing the random diffusion equation in a $D$-dimensional infinite medium ($\tau > 0$)

$$\frac{\partial P(x, \tau)}{\partial \tau} = \gamma \frac{\partial^2}{\partial x^2} P(x, \tau) + \beta V(x, \tau) P(x, \tau) \tag{1}$$

where $P(x, \tau)$ is the diffusion field, $\gamma$ the medium diffusivity constant and $V(x, \tau)$ the time-dependent random potential satisfying the Gaussian local distribution, with a positive $\beta$-coupling constant strength:

$$\langle V(x, \tau) \rangle = 0 \qquad \langle V(x_1, \tau_1) V(x_2, \tau_2) \rangle = \delta(x_1 - x_2)\delta(\tau_1 - \tau_2)$$
$$(x \in R^D) \qquad \tau \geqslant 0. \tag{2}$$

Physical quantities are functionals of the diffusion field $P(x, \tau)$ and must be averaged over the random potentials. The whole averaging information is contained in the spacetime characteristic functional (for $\tau > 0$):

$$Z[J(x, \tau)] = \left\langle \exp\left[\int_{-\infty}^{+\infty} \mathrm{d}^D x \int_0^{+\infty} \mathrm{d}\tau \, J(x, \tau) P(x, \tau, [V])\right]\right\rangle \tag{3}$$

where we have used the notation $P(x, \tau, [V])$ to emphasise that the diffusion field is a functional of the random potential $V(x, \tau)$.

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In order to write a functional integral representation for the characteristic functional we rewrite (3) as a Gaussian functional integral in $V(x, \tau)$:

$$Z[J(x,\tau)]=\int \mathscr{D}^{\mathrm{F}}[V(x,\tau)]\exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}^Dx\int_0^{\infty}\mathrm{d}\tau\, V^2(x,\tau)\right)$$
$$\times\exp\left(\int_{-\infty}^{+\infty}\mathrm{d}^Dx\int_0^{+\infty}\mathrm{d}\tau\, J(x,\tau)P(x,\tau,[V])\right). \tag{4}$$

Here the term $\mathscr{D}^{\mathrm{F}}[V(x,\tau)]$ denotes the Feynman measure ($\mathscr{D}^{\mathrm{F}}[V(x,\tau)]=\Pi_{(x\in R^D)}\Pi_{\tau\geqslant 0}\,\mathrm{d}V(x,\tau)$).

At this point we observe the validity of the functional integral representation for the $J(x,\tau)$-integrand in (4) [3].

$$\exp\left(\int_{-\infty}^{+\infty}\mathrm{d}^Dx\int_0^{+\infty}\mathrm{d}\tau\, J(x,\tau)P(x,\tau,[V])\right)$$
$$=\int \mathscr{D}^{\mathrm{F}}[w(x,\tau)]\delta_{\mathrm{F}}\left[\left(\frac{\partial}{\partial\tau}-\gamma\Delta_x-\beta V(x,\tau)\right)w(x,\tau)\right]$$
$$\times\det{}_{\mathrm{F}}\left(\frac{\partial}{\partial\tau}-\gamma\Delta_x-\beta V(x,\tau)\right)$$
$$\times\left(\exp\int_{-\infty}^{+\infty}\mathrm{d}^Dx\int_0^{+\infty}\mathrm{d}\tau\, J(x,\tau)w(x,\tau)\right) \tag{5}$$

where $\delta_{\mathrm{F}}(\cdot)$ denotes the functional delta distribution.

We now observe that the functional determinant in (5) is unity as a straightforward consequence that the Green function of the operator $\partial/\partial\tau$ is the step function (see equation (40) in [3]).

By substituting (5) in (4) and using the delta-functional integral representation

$$\delta_{\mathrm{F}}\left[\left(\frac{\partial}{\partial\xi}-\gamma\Delta_x-\beta V(x,\xi)\right)W(x,\xi)\right]$$
$$=\int \mathscr{D}^{\mathrm{F}}\lambda(x,\xi)\exp\left\{\mathrm{i}\int_{-\infty}^{+\infty}\mathrm{d}^Dx\int_0^{+\infty}\mathrm{d}\xi\,\lambda(x,\xi)\right.$$
$$\left.\times\left[\left(\frac{\partial}{\partial\xi}-\gamma\Delta_x-\beta V(x,\xi)\right)W(x,\xi)\right]\right\} \tag{6}$$

we evaluate the $V(x,\xi)$ average and obtain (after integrating the $\lambda(x,\xi)$ auxiliary field) the non-local partial result

$$Z[J(x,\tau)]=\int \mathscr{D}^{\mathrm{F}}[w(x,\tau)]\exp\left(-\frac{1}{2\beta^2}\int_{-\infty}^{+\infty}\mathrm{d}^Dx\int_0^{+\infty}\mathrm{d}\tau\frac{[((\partial/\partial\tau)-\gamma\Delta_x)w(x,\tau)]^2}{w^2(x,\tau)}\right)$$
$$\times\exp\left(\int_{-\infty}^{+\infty}\mathrm{d}^Dx\int_0^{+\infty}\mathrm{d}\tau\, J(x,\tau)w(x,\tau)\right). \tag{7}$$

Since $w(x,\tau)$ is a possible diffusion field, it is always positive and we can rewrite it as an exponential

$$w(x,\tau)=\mathrm{e}^{-\beta\phi(x,\tau)}. \tag{8}$$

By rewriting (7) in the $\phi$-variable, we obtain our local functional integral representation for the diffusion random equation (where we have used the dimensional regularisation scheme to remove the 'tadpole' functional measure Jacobian, $\delta^{(D)}(0)\int_0^{\infty} d\tau \int_0^{+\infty} d^D x \ln \phi(x, \tau) = 0$)

$$Z[J(x,\tau)] = \int \mathscr{D}^F[\phi(x,\tau)] \exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty} d^D x \int_0^{+\infty} d\tau[(\partial_\tau\phi - \gamma\Delta\phi)^2 + V(\phi)]\right) \times \exp\left(\int_{-\infty}^{+\infty} d^D x \int_0^{+\infty} d\tau\, J(x,\tau)\, e^{-\beta\phi(x,\tau)}\right) \tag{9}$$

with

$$V(\phi) = \beta^2\gamma^2|\nabla\phi|^4 + 2\gamma\beta\partial_\tau\phi|\nabla\phi|^2 - 2\gamma\beta\Delta\phi|\nabla\phi|^2. \tag{10}$$

We can thus use a bosonic perturbative diagramatic analysis [4] for the functional integral, (9) and (10). The proposed free propagator is given by the Green function of the diffusion equation

$$\left(-\frac{\partial^2}{\partial\tau^2} + \gamma^2\frac{\partial^4}{\partial x^4} - 2\gamma\frac{\partial^3}{\partial\tau\partial x^2}\right) G(x_1, x_2; \tau) = 0$$
$$\lim_{\tau\to 0^+} G(x_1, x_2; \tau) = \delta^D(x_1 - x_2) \qquad \lim_{\tau\to\infty} G(x_1, x_2; \tau) = 0. \tag{11}$$

Its expression in $k$-momentum space is given explicitly by

$$\hat{G}(K, \tau) = \exp(-\gamma|(1-\sqrt{2})||k|^2\tau). \tag{12}$$

The interaction vertices are thus given by

$$\beta^2\gamma^2 \prod_{i=1}^{4} \int_{-\infty}^{+\infty} \frac{d^D k_i}{(2\pi)^{D/2}} \int_0^{+\infty} d\tau_i\, \delta(k_1 + k_2 + k_3 + k_4)(k_1 k_2)(k_3 k_4) \times \hat{W}(k_1, \tau_1)\hat{W}(k_2, \tau_2)\hat{W}(k_3, \tau_3)\hat{W}(k_4, \tau_4) \tag{13}$$

$$-2\gamma\beta \prod_{i=1}^{3} \int_{-\infty}^{+\infty} \frac{d^D k_i}{(2\pi)^{D/2}} \int_0^{+\infty} d\tau_i\, \delta(k_1 + k_2 + k_3)|k_3|^2(k_1 k_2) \times \hat{W}(k_1, \tau_1)\hat{W}(k_2, \tau_2)\hat{W}(k_3, \tau_3) \tag{14}$$

$$2\gamma\beta \prod_{i=1}^{3} \int_{-\infty}^{+\infty} \frac{d^D k_i}{(2\pi)^{D/2}} \int_0^{+\infty} d\tau_i\, \delta(k_1 + k_2 + k_3)(k_2 k_3) \times \partial_{\tau_1}\hat{W}(k_1, \tau_1)\hat{W}(k_2, \tau_2)\hat{W}(k_3, \tau_3). \tag{15}$$

Let us analyse (9) in a kind of mean field approximation by vanishing the $V(\phi)$ interaction potential. In this case we have the exact expression for the two-point correlation function,

$$\langle P(x_1, \tau_1; [V])P(x_2, \tau_2; [V])\rangle_{\beta\to 0^+} - 1 = \lim_{\beta\to 0^+} [\exp(-\beta^2 G(x_1, x_2, \tau_1, \tau_2)) - 1] = -\beta^2 G(x_1, x_2, \tau_1, \tau_2) \tag{16}$$

which has the following decay behaviour for $x_1 = x_2$, $\tau_2 = 0$ and $\tau_1 \to \infty$ and $x \in R^3$:

$$\langle P(0, \tau, [V])P(0, 0, [V])\rangle \underset{\tau\to\infty}{\sim} -C_2\beta^2(\gamma\tau)^{-3/2} \tag{17}$$

since

$$G(0, \tau) \underset{\tau\to\infty}{\sim} \int_0^{\infty} d\omega\, e^{-\omega\tau}\left(\int_{R^3} d^3 p \frac{1}{\omega + (1-\sqrt{2})\gamma|k|^2}\right). \tag{18}$$

The renormalisation group analysis for the Euclidian quantum field theory obtained above will be implemented elsewhere following similar studies done in [2] where we used the well known Wyld [5] perturbative expansion to obtain the phase diagram of the model (1). Finally we point out that by possessing a functional integral representation for (1), the task of implementing numerical simulation for its solution can be done in a more invariant way by using Monte Carlo techniques.

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